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Line of Sight of an Aberrated Optical System

V. N. MAHAJAN
Electronics and Optics Division
Engineering Group
The Aerospace Corporation
El Segundo, Calif. 90245

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David V Thyfault

D. V. Thyfault, Captain, USAF
Space Weapons Development Mgr
Strategic Defense Weapons SPO

John W Dettmer

J. W. Dettmer, Colonel, USAF
Director
Strategic Defense Weapons SPO

FOR THE COMMANDER

William H Crabtree

W. H. Crabtree, Colonel, USAF
Deputy Commander for Space Systems

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The line of sight (LOS) of an aberrated optical system is defined in terms of the centroid of its point spread function (PSF) and ~~It is~~ also ~~expressed~~ in terms of its optical transfer function as well as its pupil function. Although the PSFs obtained according to wave diffraction optics and ray geometrical optics

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are quite different, ~~from each other,~~ they have the same centroid. For an aberration-free pupil, different amplitude distributions across it give the same centroid location; for an aberrated pupil, not only the phase but also the amplitude distribution affects the centroid location. If the amplitude across a pupil is uniform, the centroid may be obtained from the aberration along its perimeter only, without regard for the aberration across its interior regardless of its shape. Next, an optical system with aberrated but uniformly illuminated annular pupil is considered. The aberration function is expanded in terms of Zernike annular polynomials. It is shown that only those aberrations that vary with angle as $\cos\theta$ or $\sin\theta$ contribute to the ~~line of sight.~~ LOS. A simple expression is obtained for the ~~line of sight~~ LOS in terms of the Zernike aberration coefficients. Similar results are obtained for annular pupils with radially symmetric illumination. ~~Finally,~~ specific results are discussed for annular pupils aberrated by classical primary and secondary coma. As an example of a radially symmetric illumination, we obtain numerical results for Gaussian illumination of aberrated annular pupils. It is emphasized that the centroid and the peak of the PSFs aberrated by coma are not coincident, and ~~Moreover,~~ as the amount of the aberration increases, ~~the~~ separation of the centroid and peak locations also increases.

CONTENTS

FIGURES

TABLES

I. INTRODUCTION

The line-of-sight (LOS) of an aberrated optical system is defined in terms of the centroid of its point spread function (PSF). Using the Fourier transform relationship between the PSF and the optical transfer function (OTF), it is also expressed as the slope of the imaginary part of the OTF at the origin. Since the system OTF is equal to the autocorrelation of its pupil function, the centroid can also be written in terms of the amplitude and aberration across its pupil. Using the expression in terms of the pupil function, it is easy to show that the LOS obtained by wave diffraction optics is identical with that based on geometrical optics. It is shown that, whereas for an aberration-free pupil, different amplitude distributions across it give the same LOS, regardless of its shape; for an aberrated pupil both the aberration and the amplitude distribution affect the LOS. For a uniform amplitude distribution, the LOS depends only upon the aberration along the boundary of the pupil, i.e., it is independent of the aberration across its interior.

Next, an optical system with aberrated but uniformly illuminated annular pupil is considered. Its aberration function is expanded in terms of Zernike annular polynomials. It is shown that only those aberrations contribute to the LOS that vary with angle as $\cos\theta$ or $\sin\theta$, i.e., various orders of coma-type aberrations. A simple expression is obtained for the LOS in terms of the Zernike aberration coefficients. It is shown that two different orders of Zernike coma (including tilts) with the same standard deviation across the pupil do not contribute the same amount of LOS error; for a given standard deviation, a higher-order Zernike coma gives a larger LOS error. Finally, some numerical examples are considered. In particular, the PSF's aberrated by primary and secondary classical coma are discussed. It is shown that such aberrations shift the peak and centroid of the PSF (which are coincident in the aberration-free case) by significantly different amounts. Similar results are obtained for annular pupils with radially symmetric illumination. These results are illustrated by considering Gaussian illumination of an aberrated annular pupil. Numerical results for both circular and annular pupils are also given. In particular, it is shown that two PSF's aberrated by the same amount of primary and secondary coma have identical centroids in the case of uniform illumination, but different centroids for Gaussian illumination.

The results obtained here are applicable to both imaging systems as well as laser transmitters. For a certain point object, the amplitude distribution across the pupil of an imaging system will generally be uniform. However, across a laser beam, the amplitude distribution is often given by a Gaussian function. Moreover, laser beams are often polarized. If the polarization is linear, then the scalar treatment of diffraction considered in this paper leads to a linearly polarized diffracted wave. If the beam is circularly or elliptically polarized, then the diffraction of the two orthogonal polarization components may be treated separately. The total diffracted field is obtained by taking a vector sum of the two diffracted components.

II. THEORY

The PSF of an optical imaging system, i.e., the irradiance distribution of the image of a certain point object, according to wave diffraction optics is given by[1]

$$I(x,y) = (1/\lambda^2 R^2) \left| \iint_S P(u,v) \exp[-2\pi i(xu + yv)/\lambda R] \, du \, dv \right|^2, \tag{1a}$$

where λ is the wavelength of the object radiation and $i = \sqrt{-1}$. $P(u,v)$ is the pupil function of the system corresponding to this point object, and if $A(u,v)$ and $W(u,v)$ represent the amplitude and aberration of the light wave at a point (u,v) on its exit pupil, then

$$\begin{aligned} P(u,v) &= A(u,v) \exp[2\pi i W(u,v)/\lambda], \text{ inside the pupil,} \\ &= 0, \text{ outside the pupil.} \end{aligned} \tag{1b}$$

The integration in Eq. (1a) is carried out over the clear or illuminated region of the pupil. S represents the area of this region.

The aberration function $W(u,v)$ represents the deviation of the optical wavefront at the exit pupil from a spherical wavefront, called the reference sphere, measured along a ray passing through the point (u,v). The aberration $W(u,v)$ is considered positive if a ray from the point object passing through the point (u,v) has to travel a longer optical path in reaching the reference sphere than a reference ray which passes through the center of the exit pupil. The optical wavefront and the reference sphere pass through the center of the exit pupil. The reference sphere has a radius of curvature R. Its center of curvature is generally chosen to be at the Gaussian image of the point object. The center of curvature may also be chosen such that the variance of $W(u,v)$ across the pupil is minimized. In either case, the center of curvature defines the origin of the (x,y) image plane, which is parallel to the (u,v) plane. The line joining the centers of the exit pupil and the reference sphere defines a reference axis. In a system with an axis of rotational symmetry, it may be considered as the reference axis.

The LOS for the point object as perceived by the optical system is determined by the centroid of its PSF. If $\langle x \rangle$ and $\langle y \rangle$ represent the coordinates of this centroid, then

$$\langle x \rangle = E^{-1} \iint_{-\infty}^{\infty} x \, I(x,y) \, dx \, dy, \tag{2a}$$

and

$$\langle y \rangle = E^{-1} \iint_{-\infty}^{\infty} y \, I(x,y) \, dx \, dy, \tag{2b}$$

where

$$E = \iint_{-\infty}^{\infty} I(x,y) \, dx \, dy, \tag{3a}$$

is the total power (energy) in the image. Applying Parseval's theorem to Eq. (1), we find that E is also given by

$$\begin{aligned} E &= \iint_S |P(u,v)|^2 \, du \, dv \\ &= \iint_S I(u,v) \, du \, dv \, , \end{aligned} \tag{3b}$$

where

$$I(u,v) = A^2(u,v) \tag{4}$$

is the irradiance at a pupil point (u,v). Equations (3a) and (3b) represent conservation of energy; i.e., the total energy in the image is equal to the total energy in the pupil. If $\langle\alpha\rangle$ and $\langle\beta\rangle$ represent the angular LOS, they are given by

$$\langle\alpha\rangle = \langle x\rangle/R \tag{5a}$$

and

$$\langle\beta\rangle = \langle y\rangle/R. \tag{5b}$$

where we have assumed that the angles are small so that they are approximately equal to their tangents.

The OTF of an imaging system is equal to the Fourier transform of its PSF. Thus if $\tau(\xi,\eta)$ represents the OTF corresponding to a spatial frequency (ξ,η), then

$$\tau(\xi,\eta) = E^{-1} \iint_{-\infty}^{\infty} I(x,y)\exp[2\pi i(\xi x + \eta y)]\, dx\, dy. \tag{6}$$

Differentiating both sides of Eq. (6) with respect to ξ and evaluating the result at $\xi = \eta = 0$, we find that

$$\langle x\rangle = \frac{1}{2\pi i} \left(\frac{\partial\tau}{\partial\xi}\right)_{\xi=\eta=0}. \tag{7a}$$

Similarly,

$$\langle y\rangle = \frac{1}{2\pi i} \left(\frac{\partial\tau}{\partial\eta}\right)_{\xi=\eta=0}. \tag{7b}$$

Thus the centroid of the PSF of an optical system is given by the slope of its corresponding OTF at the origin. However, since $\langle x\rangle$ and $\langle y\rangle$ are real, only the slope of the imaginary part of the OTF at the origin contributes to the centroid.[2] Thus, we may write

$$\langle x\rangle = \frac{1}{2\pi}\left(\frac{\partial \mathrm{Im}\tau}{\partial \xi}\right)_{\xi=\eta=0}, \tag{8a}$$

and

$$\langle y\rangle = \frac{1}{2\pi}\left(\frac{\partial \mathrm{Im}\tau}{\partial \eta}\right)_{\xi=\eta=0}. \tag{8b}$$

The OTF is also given by the autocorrelation of the pupil function as may be seen by substituting Eq. (1) into Eq. (6).
Thus,

$$\tau(\xi,\eta) = E^{-1} \iint_{\Sigma} P(u,v)\, P^*(u - \lambda R\xi, v - \lambda R\eta)\, du\, dv, \tag{9}$$

where * indicates a complex conjugate, and Σ is the region of overlap of two pupils centered at (0,0) and $(\lambda R\xi, \lambda R\eta)$. Substituting Eq. (9) into Eq. (8), we obtain[3]

$$\langle x\rangle = -(\lambda R/2\pi E) \iint_{S} \mathrm{Im}\left[P(u,v)\frac{\partial P^*(u,v)}{\partial u}\right] du\, dv\ , \tag{10}$$

and a similar equation for $\langle y\rangle$. Substituting Eq. (1b) into Eq. (10), we obtain

$$\langle x\rangle = \frac{R}{E} \iint_{S} I(u,v)\, \frac{\partial W(u,v)}{\partial u}\ du\, dv\ . \tag{11a}$$

Similarly,

$$\langle y\rangle = \frac{R}{E} \iint_{S} I(u,v)\, \frac{\partial W(u,v)}{\partial v}\ du\, dv\ . \tag{11b}$$

Since $R(\partial W/\partial u)$ and $R(\partial W/\partial v)$ represent the ray aberrations;[4] i.e., the image-plane coordinates of a ray passing through the pupil point (u,v), Eq. (11) shows that the centroid of the PSF according to wave diffraction optics is identical with that according to ray geometrical optics.[5] (The PSF given by ray geometrical optics is called the spot diagram).

From Eq. (11), we also note that amplitude variations across the pupil affect the LOS only if it is aberrated. In the absence of aberrations, the PSF centroid lies at the center of curvature of the reference sphere regardless of the shape of the pupil and the amplitude distribution across it. This may also be seen from Eqs. (1) and (3). From Eq. (1), we note that if $W(u,v) = 0$, then $I(x,y) = I(-x,-y)$. Hence the symmetry of the aberration-free PSF yields its centroid at the origin of the (x,y) image plane. Similarly, since the aberration-free OTF is real (see Eq. (9)), Eq. (8) also gives the centroid at the origin.

It should be noted that the peak value of an aberrated PSF may or may not lie at its origin, depending on the magnitude and the type of aberration, whether or not the amplitude across the pupil is uniform. However, the peak value of an aberration-free PSF always lies at its origin, regardless of the amplitude variation across the pupil. This may be seen from Eq. (1). If we let

$$f(u,v) = P(u,v) \exp[-2\pi i(xu + yv)/\lambda R], \tag{12a}$$

then, using Hölder's inequality, we find that

$$\begin{aligned} \frac{I(x,y)}{I(0,0)} &= \frac{\left| \iint_S f(u,v)\, du\, dv \right|^2}{\iint_S |f(u,v)|^2 du\, dv} \\ &\leq 1 . \end{aligned} \tag{12b}$$

Thus, both the centroid and the peak value of an aberration-free PSF lie at its origin, regardless of the amplitude variations across the pupil.

Equations (3), (8), and (11) give the LOS of the system in terms of its PSF, OTF, and the aberration function. In practice, given an imaging system, the most convenient expression to use would be Eq. (3), since the PSF can be measured easily by using a photodetector array. In optical design and analysis, the simplest way to obtain the LOS would be to use Eq. (11), since the aberrations must be calculated even if the other two expressions were used. Thus one may trace rays all the way up to the image plane and determine the centroid of the ray distribution in this plane with appropriate weighting $I(u,v)$ of each ray.

If the illumination across the pupil is uniform, e.g., if

$$A(u,v) = A_0 , \tag{13a}$$

and

$$\begin{aligned} I(u,v) &= A_0^2 \\ &= I_0 , \end{aligned} \tag{13b}$$

and, therefore,

$$E = SI_0 , \tag{13c}$$

then Eq. (11) reduces to

$$\langle x \rangle = \frac{R}{S} \iint_S \frac{\partial W(u,v)}{\partial u} \, du \, dv \tag{14a}$$

and

$$\langle y \rangle = \frac{R}{S} \iint_S \frac{\partial W(u,v)}{\partial v} \, du \, dv . \tag{14b}$$

Using Stokes theorem, the surface integral in Eq. (14) involving the derivative of the aberration function can be written in terms of its line integral along the curve bounding the surface. Thus, we may write[6]

$$\langle x \rangle = (R/S) \oint W(u,v) \, \hat{u} \cdot d\vec{s} \tag{15a}$$

and

$$\langle y \rangle = (R/S) \oint W(u,v) \, \hat{v} \cdot d\vec{s} , \tag{15b}$$

where $\hat{u}$ and $\hat{v}$ are unit vectors along the u and v axes, respectively, $d\vec{s}$ represents an element of arc length vector of the curve bounding the pupil. It is evident from Eq. (15) that in the case of an aberrated but uniformly illuminated pupil, the centroid of the PSF can be obtained from the value of the aberration function only along the perimeter of the pupil. Accordingly, in that case, to calculate the centroid the knowledge of the aberration across the interior of the pupil is not needed.

III. APPLICATION TO SYSTEMS WITH ANNULAR PUPILS

A. Uniform Illumination

Consider an imaging system with an aberrated but uniformly illuminated annular pupil of inner and outer radii of εa and a, respectively, where $0 \leq \varepsilon \leq 1$. The area of the pupil is given by

$$S = \pi(1-\varepsilon^2)a^2 . \tag{16}$$

Owing to the circular boundary of the pupil, it is convenient to use plane polar coordinates (h,θ), where

$$u = h \cos\theta \tag{17a}$$

and

$$v = h \sin\theta, \tag{17b}$$

so that

$$\theta = \tan^{-1}(v/u) \tag{17c}$$

and

$$h = (u^2 + v^2)^{1/2}. \tag{17d}$$

Moreover, $0 \leq \theta < 2\pi$ and $\varepsilon a \leq h \leq a$. A vector element of a circular arc with a center of curvature at (0,0) and passing through a point (u,v) is given by

$$\vec{ds} = (u\, d\theta,\ v\, d\theta), \tag{18}$$

Let the aberration function in polar coordinates be $W(h,\theta;\varepsilon)$. Substituting Eq. (18) into Eq. (15), we obtain

$$\langle x \rangle = [R/\pi(1-\varepsilon^2)a] \int_0^{2\pi} [W(a,\theta;\varepsilon) - \varepsilon\, W(\varepsilon a,\theta;\varepsilon)] \cos\theta\, d\theta \tag{19a}$$

and

$$\langle y \rangle = [R/\pi(1-\varepsilon^2)a] \int_0^{2\pi} [W(a,\theta;\varepsilon) - \varepsilon\, W(\varepsilon a,\theta;\varepsilon)] \sin\theta \, d\theta \, . \tag{19b}$$

Let us expand the aberration function $W(h,\theta;\varepsilon)$ in terms of a complete set of Zernike annular polynomials $R_n^m(\rho;\varepsilon) \cos m\theta$ and $R_n^m(\rho;\varepsilon) \sin m\theta$ which are orthogonal over the annular pupil[7], where

$$\rho = h/a. \tag{20}$$

Thus, we may write

$$W(h,\theta;\varepsilon) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \sqrt{2(n+1)} \; R_n^m(\rho;\varepsilon) \, (c_{nm} \cos m\theta + s_{nm} \sin m\theta), \tag{21}$$

where n and m are positive integers (including zero), $n-m \geq 0$ and even,

$$\begin{aligned} \varepsilon_m &= 1/\sqrt{2}\,, \quad m = 0 \\ &= 1 \quad\;\;\;, \quad m \neq 0, \end{aligned} \tag{22}$$

and c_{nm} and s_{nm} are the expansion coefficients. Note that

$$s_{n0} = 0. \tag{23}$$

The radial polynomials obey the orthogonality relation

$$\int_\varepsilon^1 R_n^m(\rho;\varepsilon) \, R_{n'}^m(\rho;\varepsilon) \, \rho \, d\rho = \frac{(1-\varepsilon^2)}{2(n+1)} \, \delta_{nn'} \, , \tag{24}$$

where $\delta_{nn'}$ is a Kronecker delta. The magnitude of each expansion coefficient (except c_{00}) represents the standard deviation of the corresponding aberration term across the annular pupil.

Substituting Eq. (21) into Eq. (19), and noting the orthogonality of trigonometric functions, we obtain

$$\langle x\rangle = [R/(1-\varepsilon^2)a] \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)} \quad [R_n^1(1;\varepsilon) - \varepsilon\, R_n^1(\varepsilon;\varepsilon)]\, c_{n1} \tag{25a}$$

and

$$\langle y\rangle = [R/(1-\varepsilon^2)a] \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)} \quad [R_n^1(1;\varepsilon) - \varepsilon\, R_n^1(\varepsilon;\varepsilon)]\, s_{n1}, \tag{25b}$$

where a prime on the summation sign indicates a summation over odd integral values of n. Thus the only aberrations that contribute to the LOS error are those with $m = 1$. Aberrations of the type $R_n^1(\rho;\varepsilon)\cos\theta$ contribute to $\langle x\rangle$ and those of the type $R_n^1(\rho;\varepsilon)\sin\theta$ contribute to $\langle y\rangle$. This is also evident from the symmetry of the aberrations. Since $R_n^1(\rho;\varepsilon)$ consists of terms in ρ^n, ρ^{n-2},..., and ρ, therefore, for example, $R_n^1(\rho;\varepsilon)\cos\theta$ is symmetric in v but not in u. Hence the PSF is symmetric in y as may be seen from Eq. (1) noting that the amplitude across the pupil is uniform. Accordingly, $\langle y\rangle = 0$ for this aberration.

It is well known that, for small aberrations, the Strehl ratio for an optical system depends on the variance of its aberration[8]. Therefore, two aberration terms with $m = 1$, but different values of n, affect the Strehl ratio in the same way if their coefficients are equal in magnitude. However, it is evident from Eq. (25) that their contribution to the LOS error is not the same; an aberration of higher order contributes a larger error for the same value of the coefficient. Hence, in tolerancing an optical system, one must be careful in allocating equal standard deviation to two aberration terms that also contribute to the LOS error.

In the case of a circular aperture ($\varepsilon = 0$),

$$R_n^1(1;0) = 1 \; . \tag{26}$$

Therefore, Eq. (25) simplifies to

$$\langle x\rangle = (R/a) \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)}\; c_{n1} \tag{27a}$$

and

$$\langle y\rangle = (R/a) \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)}\; s_{n1} . \tag{27b}$$

We note that as in the case of an annular pupil, two aberration terms with equal standard deviation but different order n do not contribute equally to the LOS error. For a given standard deviation, a higher-order aberration gives a larger LOS error compared to a lower-order aberration.

B. Radially Symmetric Illumination

Let A(h) and I(h) describe the radially symmetric amplitude and irradiance distributions across the aberrated annular pupil, where

$$I(h) = A^2(h). \tag{28}$$

In polar coordinates, Eq. (11) for the LOS can be written

$$\langle x\rangle = \frac{R}{E} \int_{\epsilon a}^{a} \int_{0}^{2\pi} I(h) \left[\cos\theta \; \frac{\partial W(h,\theta;\epsilon)}{\partial h} - \frac{\sin\theta}{h} \frac{\partial W(h,\theta;\epsilon)}{\partial \theta}\right] h \, dh \, d\theta \tag{29a}$$

and

$$\langle y\rangle = \frac{R}{E} \int_{\epsilon a}^{a} \int_{0}^{2\pi} I(h) \left[\sin\theta \; \frac{\partial W(h,\theta;\epsilon)}{\partial h} + \frac{\cos\theta}{h} \frac{\partial W(h,\theta;\epsilon)}{\partial \theta}\right] h \, dh \, d\theta, \tag{29b}$$

where

$$E = 2\pi \int_{\epsilon a}^{a} I(h) \, h \, dh \, d\theta . \tag{30}$$

We now expand the aberration function in terms of annular polynomials $S_n^m(\rho;\varepsilon)\cos m\theta$ and $S_n^m(\rho;\varepsilon)\sin m\theta$ that are orthogonal over the radially symmetric illuminated annular pupil[7]. Thus we write

$$W(h,\theta;\varepsilon) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \sqrt{2(n+1)}\; S_n^m(\rho;\varepsilon)(c_{nm}\cos m\theta + s_{nm}\sin m\theta), \quad (31)$$

where

$$S_n^m(\rho;\varepsilon) = M_n^m \left[R_n^m(\rho;\varepsilon) - \sum_{i\geq i}^{(n-m)/2} (n-2i+1)\langle R_n^m(\rho;\varepsilon)\; S_{n-2i}^m(\rho;\varepsilon)\rangle\; S_{n-2i}^m(\rho;\varepsilon) \right] \quad (32)$$

is a radial polynomial with properties similar to those of $R_n^m(\rho;\varepsilon)$. It should be evident that the aberration coefficients c_{nm} and s_{nm} in Eq. (31) are different from those in Eq. (21). In Eq. (32)

$$\langle R_n^m(\rho;\varepsilon)\; S_{n-2i}^m(\rho;\varepsilon)\rangle = \int_\varepsilon^1 \int_0^{2\pi} R_n^m(\rho;\varepsilon)\; S_{n-2i}^m(\rho;\varepsilon)\; A(a\rho)\rho\, d\rho \Big/ \int_\varepsilon^1 \int_0^{2\pi} A(a\rho)\;\rho\; d\rho, \quad (33)$$

and M_n^m is a normalization constant such that

$$\int_\varepsilon^1 S_n^m(\rho;\varepsilon)\; S_{n'}^m(\rho;\varepsilon)\; A(\rho)\;\rho\; d\rho \Big/ \int_\varepsilon^1 A(\rho)\;\rho\; d\rho = \frac{1}{n+1}\,\delta_{nn'} \;. \quad (34)$$

Substituting Eq. (31) into Eq. (29) we find that

$$\langle x\rangle = \frac{\pi a R}{E} \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)}\;\; c_{n1} \int_\varepsilon^1 I(a\rho)\; \frac{\partial}{\partial\rho}\,[\rho\, S_n^1(\rho;\varepsilon)]\; d\rho \quad (35a)$$

and

$$\langle y\rangle = \frac{\pi a R}{E} \sum_{n=1}^{\infty}{}' \sqrt{2(n+1)}\;\; s_{n1} \int_\varepsilon^1 I(a\rho)\; \frac{\partial}{\partial\rho}\,[\rho\, S_n^1(\rho;\varepsilon)]\; d\rho. \quad (35b)$$

Once again the only aberrations that contribute to the LOS error are those with $m = 1$. Aberrations of the type $S_n^1(\rho;\varepsilon)\cos\theta$ contribute to $\langle x \rangle$ and those of the type $S_n^1(\rho;\varepsilon)\sin\theta$ contribute to $\langle y \rangle$, and this can be explained from the symmetry properties of the aberrated PSF in a manner similar to the discussion following Eq. (25).

IV. NUMERICAL EXAMPLES

A. Uniform Illumination

Using polar coordinates

$$x = r\cos\phi \tag{36a}$$

and

$$y = r\sin\phi, \tag{36b}$$

it can be shown that for an aberration-free optical system with a uniformly illuminated annular pupil, Eq. (1) reduces to[9]

$$I(r;\varepsilon) = \frac{1}{(1-\varepsilon^2)^2}\left[\frac{2J_1(\pi r_s)}{\pi r_s} - \varepsilon^2\,\frac{2J_1(\pi\varepsilon r_s)}{\pi\varepsilon r_s}\right]^2 I(0;\varepsilon). \tag{37}$$

Here

$$r = (x^2 + y^2)^{1/2} \tag{38}$$

is the radial distance of a point (x,y) in the image plane from its origin, and

$$r_s = r/\lambda F \tag{39}$$

is a scaled radial distance of an image point. $J_1(\cdot)$ is the first-order Bessel function of the first kind, and

$$F = R/2a \tag{40}$$

is the f-number of the system. The central value of the PSF is given by

$$I(0;\epsilon) = ES/\lambda^2R^2, \tag{41}$$

where the power E in the image is given by Eq. (13c).

The line joining the center of the pupil and the centroid of the PSF for a given point object defines the LOS of the optical system in its image space for that point object. Since the aberration-free PSF is circularly symmetric about the origin of the (x,y) plane, its centroid also lies at this origin, i.e., it lies at the center of curvature of the reference sphere with respect to which the aberration for the point object under consideration is zero. Hence, the line joining the origins of the (u,v) and (x,y) planes defines the LOS.

When aberrations are introduced into the system such that the centroid of its PSF shifts, the position of the point object as perceived by the system changes, i.e., there is a LOS error. We have shown that in the case of optical systems with pupils having circular boundaries and uniform or radially symmetric amplitude distributions, the only aberrations that contribute to the LOS error are of the type $R_n^1(\rho;\epsilon)\cos\theta$ and $R_n^1(\rho;\epsilon)\sin\theta$. Since the radial polynomials $R_n^1(\rho;\epsilon)$ consist of terms in ρ^n, ρ^{n-2},..., and ρ, with their coefficients varying with ϵ, there is no loss of generality if we consider aberrations of the type $\rho^n\cos\theta$ (or $\rho^n\sin\theta$), where n is an odd integer, to determine their contribution to the LOS error. Thus, we consider an aberration

$$W(h,\theta) = W_n\,(h/a)^n\cos\theta, \quad \epsilon a \le h \le a \tag{42a}$$

$$= W_n\,\rho^n\cos\theta, \qquad \epsilon \le \rho \le 1, \tag{42b}$$

where W_n is the peak aberration at the edge of the pupil relative to a value of zero at its center. Since $\langle y\rangle$ and correspondingly $\langle\beta\rangle$ are both zero for this type of an aberration, we shall not explicitly so state from now on. We shall, for example, refer to the centroid as simply $\langle x\rangle$.

Substituting Eq. (42) into Eq. (20), we obtain

$$\langle x \rangle = 2W_n F \sum_{i=0}^{(n-1)/2} \varepsilon^{2i}. \tag{43}$$

The corresponding angular LOS error is given by

$$\langle \alpha \rangle = 2(W_n/D) \sum_{i=0}^{(n-1)/2} \varepsilon^{2i}, \tag{44}$$

where

$$D = 2a \tag{45}$$

is the outer diameter of the annular pupil. We note from Eq. (44), for example, that for circular pupils ($\varepsilon = 0$), one wave ($W_n = 1\lambda$) of aberration of the type given by Eq. (42) produces an angular LOS error of $2\lambda/D$. This is quite large considering that the angular radius of the (aberration-free) Airy disc is only $1.22\ \lambda/D$.

It is interesting to note that when $\varepsilon = 0$, the LOS error depends only on the value of W_n but not on n, the power of ρ in Eq. (42b). This is consistent with our earlier observation [following Eq. (15)] that, for a uniformly illuminated pupil, the centroid of its aberrated PSF depends only on the aberration along its perimeter. In the case of a circular pupil, the variation of the aberration along its perimeter as given by Eq. (42) is the same for different orders of the aberration. Hence, for a given value of W_n, even though aberrations such as tilt ($n = 1$), primary coma ($n = 3$), secondary coma ($n = 5$), etc., which are completely different from each other across the interior of the pupil and, therefore, give completely different PSF's, nevertheless give the same centroid since the aberrations are identical on the perimeter of the pupil. This is not true for an annular pupil, in which case, although the aberration along the outer perimeter is the same for different values of n, it is different along the inner perimeter. Hence, for a given value of W_n, an annular pupil gives aberrated PSF's with different centroids for different orders of the aberration.

We may add that similar observations hold for balanced aberrations represented by Zernike polynomials $R_n^1(\rho;\epsilon)\cos\theta$ and $R_n^1(\rho;\epsilon)\sin\theta$. For example, for a given value of W_n, aberrations $W_n\, R_n^1(\rho;0)\cos\theta$ across a circular pupil give PSF's with the same centroid since $R_n^1(1;0) = 1$, i.e., since the aberration along the perimeter of the pupil is independent of the aberration order n. In the case of an annular pupil, $R_n^1(1;\epsilon)$ as well as $R_n^1(\epsilon;\epsilon)$ depend on the value of n. Hence, PSF's with different centroids are obtained for different aberration orders n.

Wavefront Tilt

If we let n = 1 in Eq. (42), the aberration is simply a tilt of the optical wavefront through the center of the pupil. The PSF shifts and its centroid moves from (0,0) to

$$\langle x\rangle = 2W_1F \tag{46a}$$

corresponding to

$$\langle\alpha\rangle = 2W_1/D. \tag{46b}$$

Thus, for example, one wave of wavefront tilt, produces an angular LOS error of $2\lambda/D$, regardless of the value of ϵ. Of course, in this case, the position of the peak value of the PSF is coincident with the position of its centroid.

Primary Coma

If we let n = 3, in Eq. (42) the aberration obtained is called classical primary coma. The centroid in this case is given by

$$\langle x\rangle = 2W_3F(1 + \epsilon^2). \tag{47}$$

Thus, for a given value of W_3, the centroid for an annular pupil shifts by a factor of $(1 + \epsilon^2)$ larger than that for a circular pupil.

For small values of W_3, the peak value of the aberrated PSF occurs at a point such that if the aberration is measured with respect to a reference sphere centered at this point, the variance of the aberration across the annular pupil is minimum. From the properties of the Zernike annular polynomials, we find that the polynomial $R_3^1(\rho;\varepsilon)\cos\theta$ gives the optimum combination of $\rho^3\cos\theta$ and $\rho\cos\theta$ terms leading to a minimum variance. Since[7]

$$R_3^1(\rho;\varepsilon) = \frac{3(1+\varepsilon^2)\rho^3 - 2(1+\varepsilon^2+\varepsilon^4)\rho}{(1-\varepsilon^2)[(1+\varepsilon^2)(1+4\varepsilon^2+\varepsilon^4)]^{1/2}}, \tag{48}$$

we note that, for small values of W_3, the peak value of the aberrated PSF occurs at

$$x_m = 4W_3F\,(1+\varepsilon^2+\varepsilon^4)/3(1+\varepsilon^2), \tag{49}$$

where the subscript m refers to the point corresponding to minimum aberration variance. From the form of the aberration, it is understood that $y_m = 0$. Thus, an amount W_3 of primary u-coma shifts the centroid and peak of the PSF by different amounts, the movement of the peak being $2(1+\varepsilon^2+\varepsilon^4)/3(1+\varepsilon^2)^2$ of the movement of the centroid. As an example, a circular pupil aberrated by a quarter wave of primary u-coma ($W_3 = \lambda/4$) gives an aberrated PSF with a centroid at $\langle x\rangle = \lambda F/2$ and a peak value at $x_m = \lambda F/3$.

For large values of W_3, the peak of the aberrated PSF does not occur at the point corresponding to minimum aberration variance[10]. For example, in the case of circular pupils, the peak lies approximately at the point corresponding to $W_1 = (2/3)W_3$ only when $W_3 \lesssim 0.7\lambda$. For larger values of W_3, the peak occurs closer to the origin than the point corresponding to minimum aberration variance. For $W_3 \gtrsim 1.6\lambda$, the distance of the peak from the origin does not increase monotonically, but fluctuates as W_3 increases.[11] Since, according to Eq. (47), the distance of the centroid increases linearly with W_3, it is clear that the separation between the locations of the centroid and peak increases as W_3 increases.

If we let $\phi = 0$ and $A(h) = A_0$ in Eq. (A6), we find that, along the x-axis, the PSF aberrated by primary coma of the type $W_3\rho^3\cos\theta$ can be written

$$I(x;\varepsilon) = [I(0;\varepsilon)/(1-\varepsilon^2)^2] \, [\int_{\varepsilon^2}^{1} J_0(\pi B)\, dt]^2, \tag{50}$$

where

$$B = (2tW_3 - x_s)t^{1/2}, \tag{51}$$

$$x_s = x/\lambda F, \tag{52}$$

and W_3 is in units of λ. Figure 1 shows how $I(x;\varepsilon)$ normalized by the aberration-free central value $I(0;\varepsilon)$ given by Eq. (41) varies with x for several typical values of W_3 varying from 0 to 2λ, and $\varepsilon = 0$ and $\varepsilon^2 = 0.5$. Figure 2 shows how the irradiance $I_m(W_3;\varepsilon)$ at x_m, the peak irradiance $I_p(W_3;\varepsilon)$ and the irradiance $I_c(W_3;\varepsilon)$ at $\langle x\rangle$ vary with W_3. Figure 3 shows how x_m, x_p (the point at which peak irradiance occurs), and $\langle x\rangle$ vary with W_3. The observations made above about the PSF's aberrated by primary coma are evident from these figures. Several typical values of x_m, x_p, and $\langle x\rangle$ and the corresponding irradiances I_m, I_p, and I_c are noted in Table 1, where the numbers without parentheses are for $\varepsilon = 0$ and those with parenthesis are for $\varepsilon^2 = 0.5$. The aberrated central irradiance I(0) is also given in this Table. The irradiance values I(0) and I_m are the Strehl ratios calculated for primary and balanced primary coma, respectively, in an earlier paper.[10]

Secondary Coma

If we let n = 5 in Eq. (42), the aberration obtained is called classical secondary coma. The centroid in this case is given by

$$\langle x\rangle = 2W_5F(1 + \varepsilon^2 + \varepsilon^4). \tag{53}$$

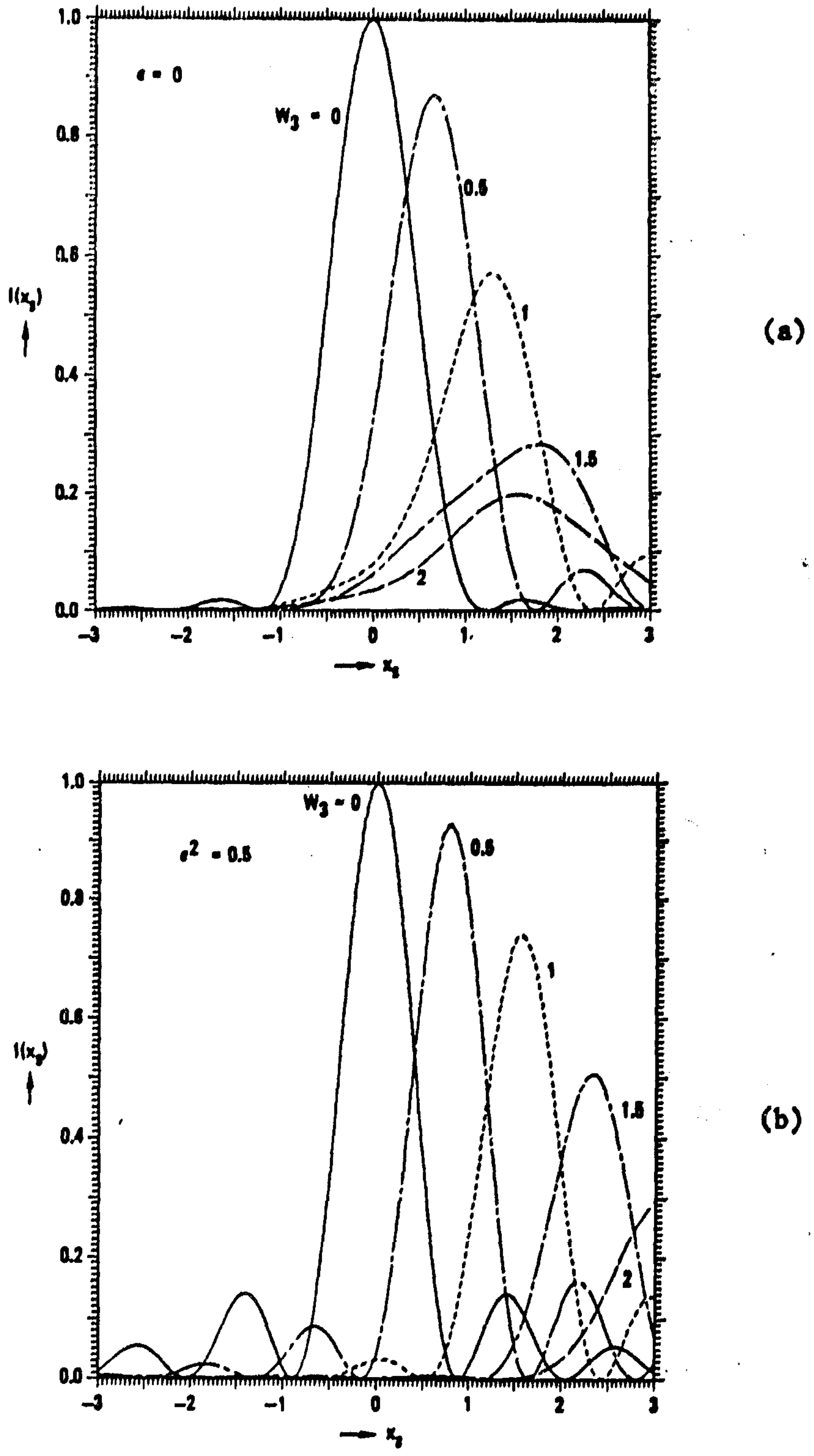
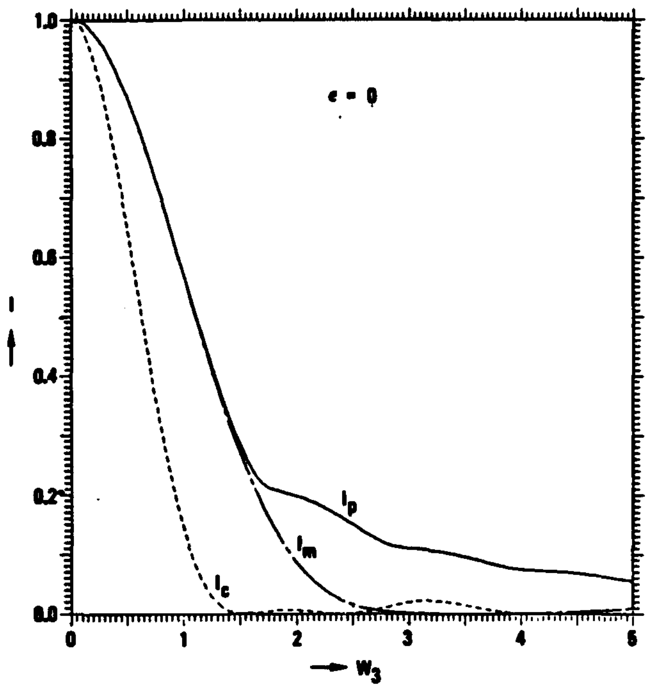
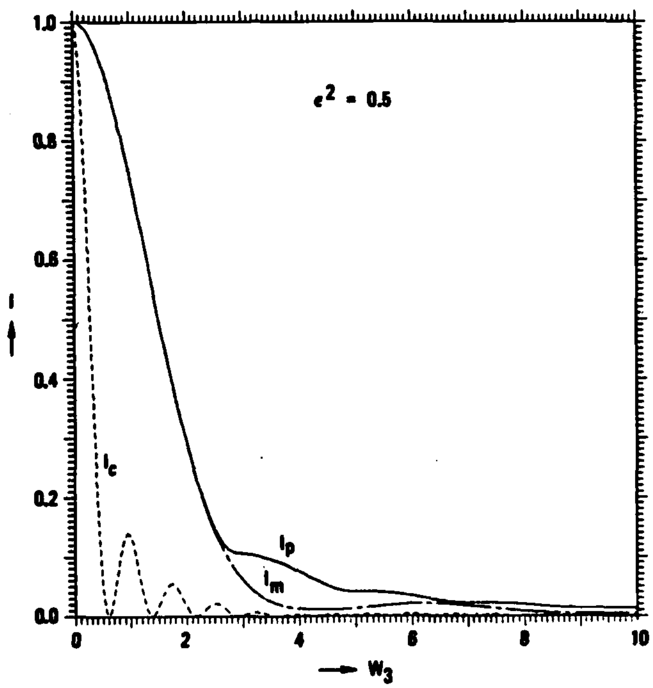


Figure 1. PSF $I(x_s;\epsilon)$ for several typical values of primary coma aberration W_3 in units of λ. The amplitude A(u,v) across the pupil is uniform. The PSF's are normalized by the aberration-free central value given by Eq. (41). x_s represents x in units of λF. (a) $\epsilon = 0$, (b) $\epsilon^2 = 0.5$.



(a)



(b)

Figure 2. Variation of I_m, I_p, and I_c with W_3, where the irradiances are in units of the aberration-free central irradiance and W_3 is in units of λ. (a) $\epsilon = 0$, (b) $\epsilon^2 = 0.5$.

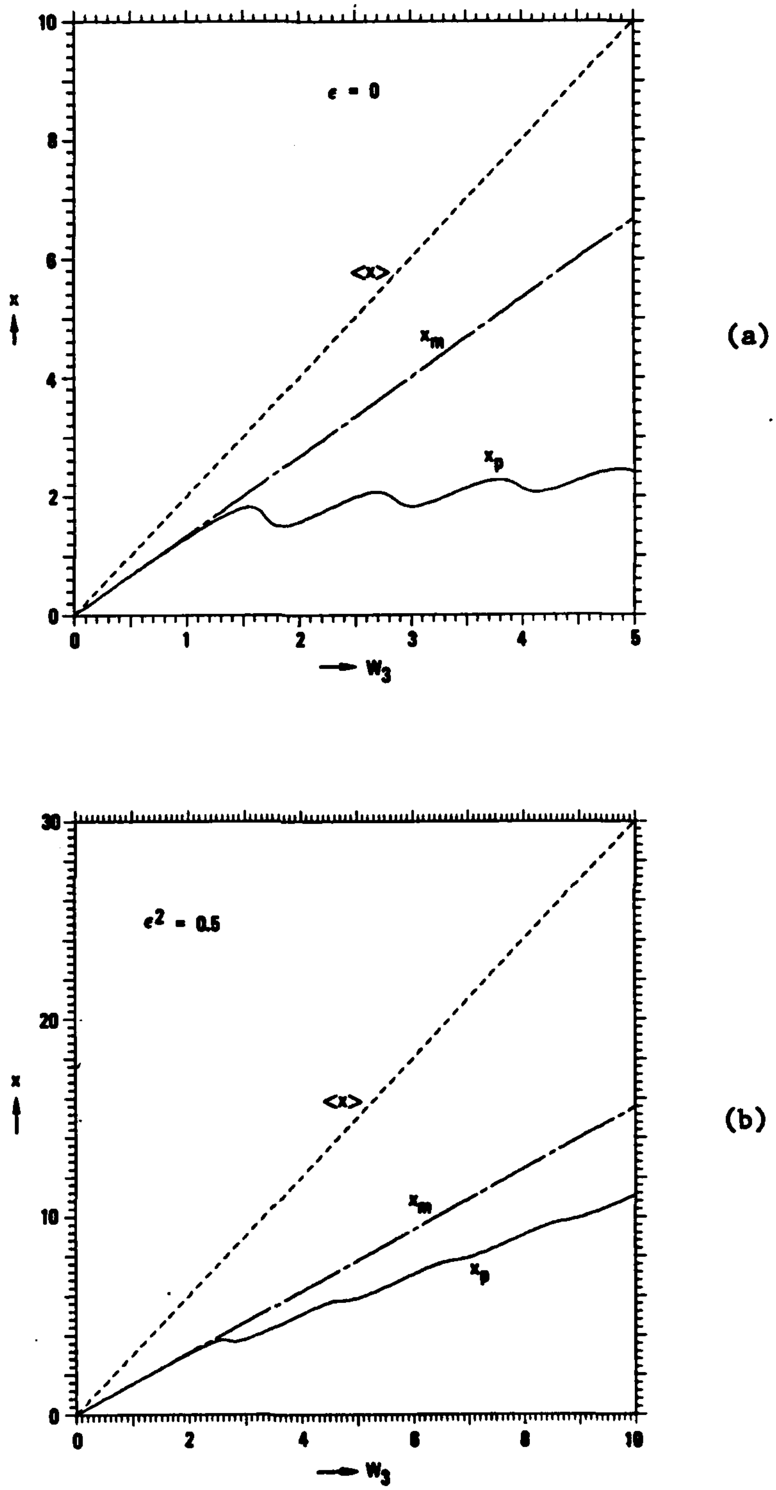


Figure 3. Variation of x_m, x_p, and $\langle x \rangle$ with W_3. The x-values are in units of λF and W_3 is in units of λ. (a) $\epsilon = 0$, (b) $\epsilon^2 = 0.5$

Table 1. Typical values of x_m, x_p, and $\langle x \rangle$ in units of λF, and the corresponding irradiances I_m, I_p, and I_c in units of the aberration-free central irradiance for PSF's aberrated by primary coma, $W(h,\theta) = W_3 \rho^3 \cos\theta$. The units of W_3 are λ. The aberrated central value $I(0)$ is also given here. The amplitude $A(u,v)$ across the pupil is uniform. The numbers without parentheses are for a circular pupil ($\varepsilon = 0$) and those with parentheses are for an annular pupil with $\varepsilon^2 = 0.5$.

W_3	x_m	x_p	$\langle x \rangle$	I_m	I_p	I_c	$I(0)$
0	0	0	0	1.0000	1.0000	1.0000	1.0000
	(0)	(0)	(0)	(1.0000)	(1.0000)	(1.0000)	(1.0000)
0.5	0.67	0.66	1.00	0.8712	0.8712	0.6535	0.3175
	(0.78)	(0.78)	(1.50)	(0.9283)	(0.9283)	(0.0524)	(0.0403)
1.0	1.33	1.30	2.00	0.5708	0.5717	0.1445	0.0791
	(1.56)	(1.55)	(3.00)	(0.7410)	(0.7412)	(0.1357)	(0.0319)
1.5	2.00	1.80	3.00	0.2715	0.2844	0.0004	0.0618
	(2.33)	(2.32)	(4.50)	(0.5063)	(0.5064)	(0.0139)	(0.0010)
2.0	2.67	1.57	4.00	0.0864	0.1978	0.0061	0.0341
	(3.11)	(3.07)	(6.00)	(0.2936)	(0.2946)	(0.0160)	(0.0000)

The variance of an aberration of the type $\rho^5\cos\theta$ is reduced if an amount $-[(1+\epsilon^2+\epsilon^4+\epsilon^6)/2(1+\epsilon^2)]W_5$ of $\rho\cos\theta$ aberration is introduced. Hence, the corresponding value of x_m is given by

$$x_m = W_5F(1+\epsilon^2+\epsilon^4+\epsilon^6)/(1+\epsilon^2). \tag{54}$$

Thus, for example, in the case of a circular pupil, the variance is reduced by a factor of 4 if $W_1 = -0.5W_5$. Accordingly, for small values of W_5, the peak of the corresponding aberrated PSF occurs at $x_m = W_5F$, while the centroid occurs at $\langle x\rangle = 2W_5F$. The aberrated PSF along the x-axis in this case is given by Eq. (50) where

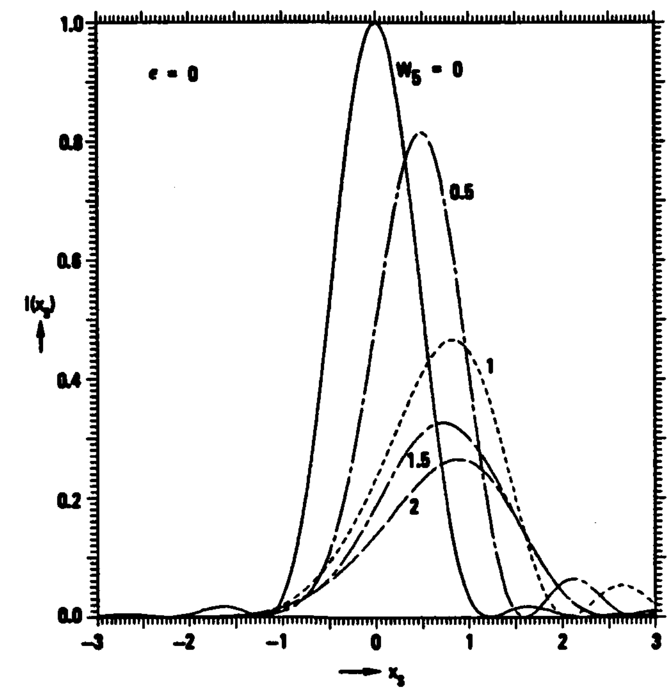
$$B = (2t^2 W_5 - x_s)t^{1/2}, \tag{55}$$

and W_5 is in units of λ. Figure 4 shows how $I(x;\epsilon)$ varies with x for several values of W_5, and $\epsilon = 0$ and $\epsilon^2 = 0.5$. The values of x_m, x_p, and $\langle x\rangle$, and the corresponding irradiances I_m, I_p, and I_c for the values of W_5 considered are noted in Table 2.

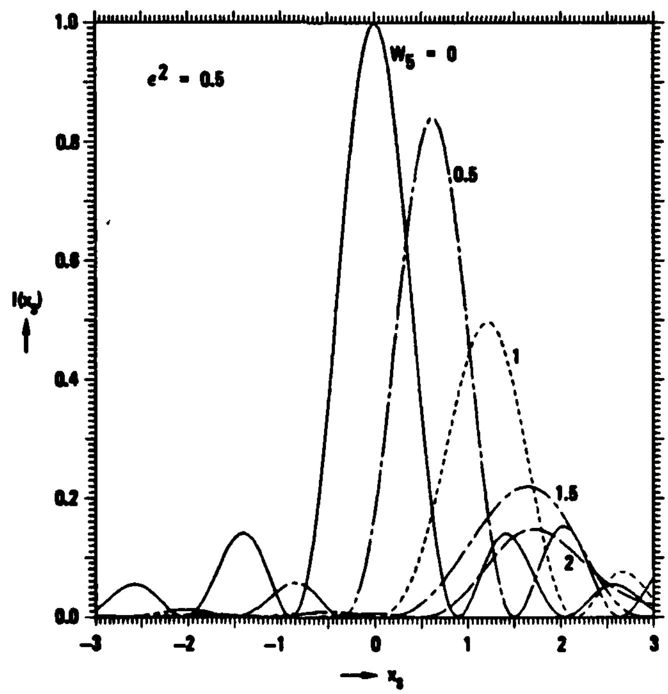
The variance of the aberration $\rho^5\cos\theta$ is reduced even further if an appropriate amount of $\rho^3\cos\theta$ aberration is also introduced. For a given value of W_5, the appropriate amounts of W_3 and W_1 that give minimum variance may be obtained from the radial Zernike annular polynomial $R_5^1(\rho;\epsilon)$, where[7]

$$R_5^1(\rho;\epsilon) = \frac{10(1+4\epsilon^2+\epsilon^4)\rho^5 - 12(1+4\epsilon^2+4\epsilon^4+\epsilon^6)\rho^3 + 3(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho}{(1-\epsilon^2)^2\,[(1+4\epsilon^2+\epsilon^4)(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]^{1/2}}. \tag{56}$$

Thus, in the case of a circular pupil, the variance is reduced by a factor of 100 if we introduce $\rho\cos\theta$ and $\rho^3\cos\theta$ aberrations with $W_1 = 0.3W_5$ and $W_3 = -1.2W_5$. Hence the peak value of the PSF for a circular pupil aberrated by a small value of W_5 and $W_3 = -1.2W_5$ occurs at $x_m = -0.6W_5F$. According to Eq. (43), the corresponding centroid occurs at $\langle x\rangle = -0.4W_5F$. Therefore, the



(a)



(b)

Figure 4. Same as Figure 1 except that the aberration is secondary coma W_5.

Table 2. Same as Table 1, except that the aberration is secondary coma, $W(h,\theta) = W_5\, \rho^5 \cos\theta$, where W_5 is in units of λ.

W_5	x_m	x_p	$\langle x \rangle$	I_m	I_p	I_c	$I(0)$
0	0	0	0	1.0000	1.0000	1.0000	1.0000
	(0)	(0)	(0)	(1.0000)	(1.0000)	(1.0000)	(1.0000)
0.5	0.50	0.49	1.00	0.8150	0.8153	0.4114	0.4955
	(0.63)	(0.62)	(1.75)	(0.8400)	(0.8402)	(0.0760)	(0.1768)
1.0	1.00	0.83	2.00	0.4464	0.4664	0.0025	0.2332
	(1.25)	(1.21)	(3.50)	(0.4948)	(0.4966)	(0.0282)	(0.0002)
1.5	1.50	0.81	3.00	0.1685	0.3237	0.0098	0.1873
	(1.88)	(1.74)	(5.25)	(0.2003)	(0.2196)	(0.0130)	(0.0009)
2.0	2.00	1.11	4.00	0.0420	0.2523	0.0073	0.1389
	(2.50)	(1.71)	(7.00)	(0.0573)	(0.1478)	(0.0074)	(0.0065)

separation between the peak and the centroid is $0.2W_5F$. For large values of W_5, minimization of variance with respect to W_3 and W_1 does not lead to a maximum of the PSF.

As an example, we consider the PSF aberrated by an aberration

$$W(h,\theta) = (W_5\rho^5 + W_3\rho^3)\cos\theta, \tag{57a}$$

where

$$W_3 = -1.2\, W_5(1+4\varepsilon^2+4\varepsilon^4+\varepsilon^6)/(1+4\varepsilon^2+\varepsilon^4). \tag{57b}$$

According to Eq. (56), the point in the image plane with respect to which the aberration variance is minimized is given by

$$x_m = -0.6\, W_5F(1+4\varepsilon^2+10\varepsilon^4+4\varepsilon^6+\varepsilon^8)/(1+4\varepsilon^2+\varepsilon^4). \tag{58}$$

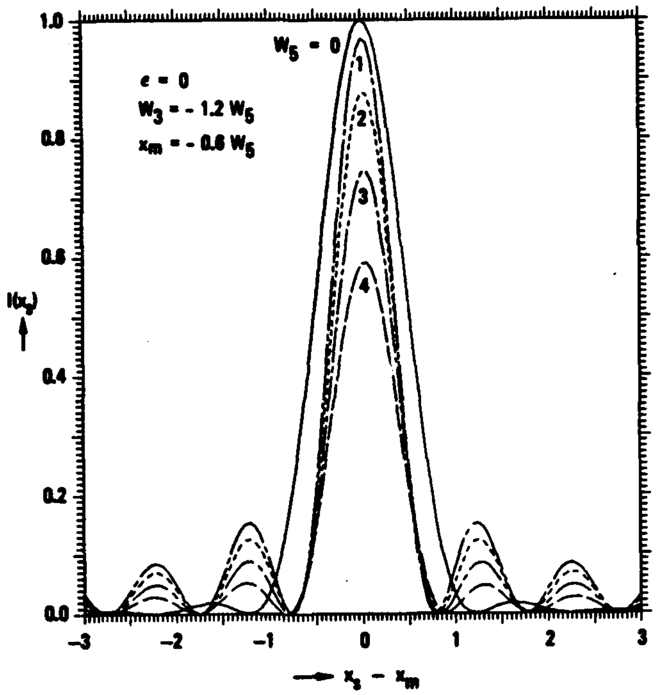
Substituting Eq. (57) into Eq. (19), we obtain the centroid

$$\langle x\rangle = -W_5F(0.4+2\varepsilon^2+7.2\varepsilon^4+2\varepsilon^6+0.4\varepsilon^8)/(1+4\varepsilon^2+\varepsilon^4). \tag{59}$$

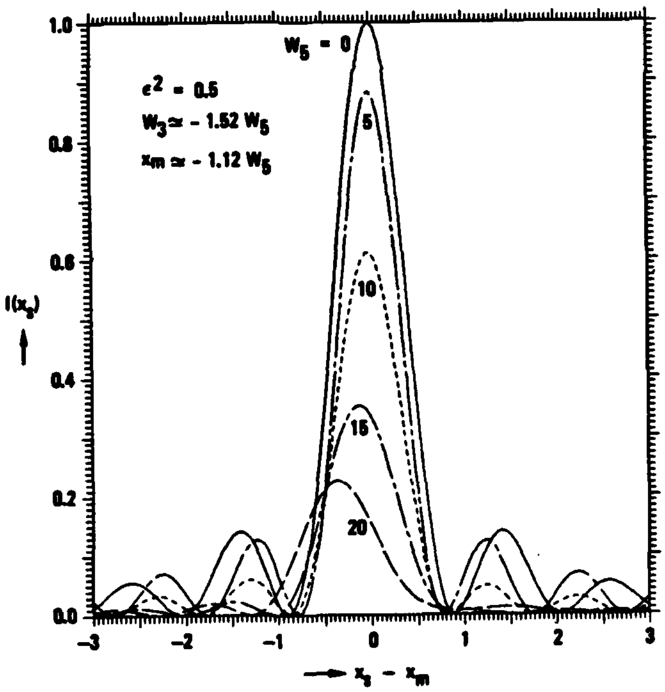
The aberrated PSF along the x-axis is obtained by substituting Eq. (57) into Eq. (A6). We find that it is given by Eq. (50), where

$$B = (2t^2 W_5 + 2tW_3 - x_s)t^{1/2}\,. \tag{60}$$

Figure 5 shows the aberrated PSF $I(x;\varepsilon)$ for several values of W_5 with W_3 given by Eq. (57), and $\varepsilon = 0$ and $\varepsilon^2 = 0.5$. The values of x_m, x_p and $\langle x\rangle$, and the corresponding irradiances I_m, I_p, I_c are given in Table 3. Note that x_m, x_p, and $\langle x\rangle$ are all negative. Moreover, their magnitude for the values of W_5 considered, especially in the case of $\varepsilon^2 = 0.5$, is very large. Therefore, in Figure 5, the horizontal coordinate is chosen to be $x_s - x_m$.



(a)



(b)

Figure 5. Same as Figure 1 except that the aberration is a combination of primary and secondary coma given by Eq. (57). Note that in this figure the horizontal coordinate is $x_s - x_m$.

Table 3. Same as Table 1, except that the aberration is a combination of primary and secondary coma given by Eq. (57).

W_5	x_m	x_p	<x>	I_m	I_p	I_c	I(0)
0	0 (0)	0 (0)	0 (0)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
1.0 (5.0)	-0.60 (-5.60)	-0.59 (-5.60)	-0.40 (-5.35)	0.9676 (0.8832)	0.9682 (0.8832)	0.8763 (0.0005)	0.3721 (0.0039)
2.0 (10.0)	-1.20 (-11.19)	-1.18 (-11.23)	-0.80 (-10.69)	0.8765 (0.6101)	0.8784 (0.6128)	0.5870 (0.0000)	0.0030 (0.0014)
3.0 (15.0)	-1.80 (-16.79)	-1.77 (-16.94)	-1.20 (-16.04)	0.7429 (0.3353)	0.7459 (0.3558)	0.2981 (0.0000)	0.0014 (0.0000)
4.0 (20.0)	-2.40 (-22.38)	-2.37 (-22.80)	-1.60 (-21.38)	0.5886 (0.1493)	0.5914 (0.2296)	0.1173 (0.0000)	0.0465 (0.0008)

B. Gaussian Illumination

As an example of a radially symmetric illumination, we consider a Gaussian annular pupil, i.e. one for which

$$A(h) = A_0 \exp[-\gamma(h/a)^2] \tag{61a}$$

$$= A_0 \exp(-\gamma\rho^2), \tag{61b}$$

where $\gamma > 0$. The aberration-free PSF for such a pupil may be obtained by substituting Eq. (61) into Eq. (1) or Eq. (A6) [Appendix A] and letting $W(h,\theta) = 0$. Thus, we obtain

$$I(r;\gamma;\epsilon) = [\gamma/(e^{-\gamma} - e^{-\gamma\epsilon^2})]^2 \, I(0,\gamma;\epsilon) \, [\int_{\epsilon^2}^{1} \exp(-\gamma t) \, J_0(\pi r_s t^{1/2}) dt]^2, \tag{62}$$

where

$$I(0;\gamma;\epsilon) = (\pi a^2 A_0^2/\lambda R)^2 \, [e^{-\gamma} - e^{-\gamma\epsilon^2})/\gamma]^2 \tag{63}$$

is the central value of the PSF.

Primary Coma

If we let $\phi = 0$ and substitute Eq. (61) into Eq. (A6), we find that, along the x-axis, the aberrated PSF in the presence of primary coma of the type $W_3\rho^3\cos\theta$ may be written

$$I(x;\gamma;\epsilon) = [\gamma/(e^{-\gamma} - e^{-\gamma\epsilon^2})]^2 \, I(0,\gamma;\epsilon) \, [\int_{\epsilon^2}^{1} \exp(-\gamma t) \, J_0(\pi B) dt]^2, \tag{64}$$

where B is given by Eq. (51).

Substituting Eq. (42) with n = 3 and Eq. (61) into Eq. (29), we find that the centroid of the PSF is given by

$$\langle x \rangle = 4 \, W_3 F \left(\frac{1}{2\gamma} + \frac{\epsilon^2 e^{-2\gamma\epsilon^2} - e^{-2\gamma}}{e^{-2\gamma\epsilon^2} - e^{-2\gamma}} \right). \tag{65}$$

From the radial polynomial $S_3^1(\rho;\epsilon)$ for the Gaussian illumination[7], we find that the point in the image plane with respect to which the aberration variance across the Gaussian pupil is minimized is given by

$$x_m = 2\, W_3 F \left[\frac{2}{\gamma} + \frac{\gamma(\epsilon^4\, e^{-\gamma\epsilon^2} - e^{-\gamma})}{e^{-\gamma\epsilon^2}(1+\gamma\epsilon^2) - e^{-\gamma}(1+\gamma)} \right]. \tag{66}$$

For small values of W_3, the peak value of the PSF occurs at x_m.

We now consider some numerical results for $\gamma = 1$, which corresponds to a Gaussian illumination with an irradiance of e^{-2} at the edge of a circular pupil relative to the irradiance at its center. Figure 6 shows how $I(x;1;\epsilon)$ varies with x for several values of W_3 and $\epsilon^2 = 0$ and 0.5. The values of x_m, x_p, and $\langle x \rangle$, and the corresponding irradiances I_m, I_p, and I_c for these values of W_3 are given in Table 4.

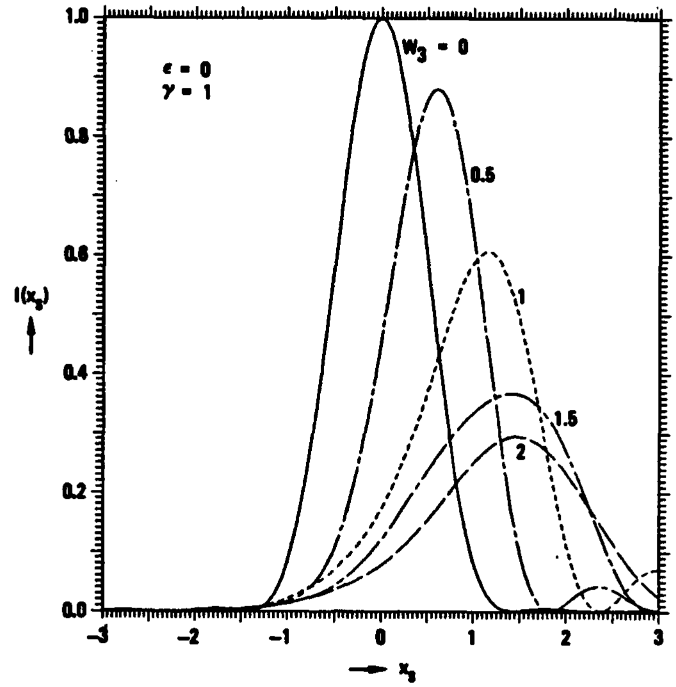
Secondary Coma

The aberrated PSF along the x-axis in the presence of secondary coma of the type $W_5\, \rho^5 \cos\theta$ is given by Eq. (64) where B is given by Eq. (55). Substituting Eq. (42) with n = 5 and Eq. (61) into Eq. (29), we find that the centroid of the aberrated PSF is given by

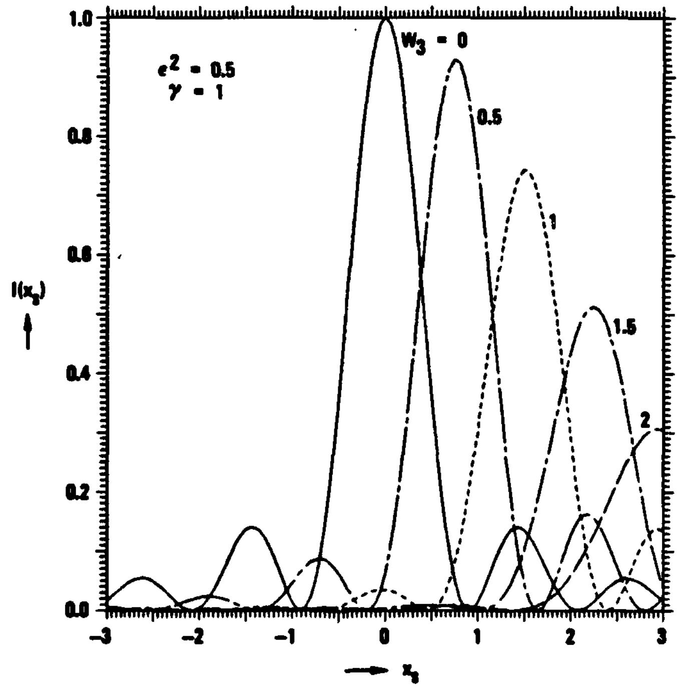
$$\langle x \rangle = 12 W_5 F\, \frac{e^{-2\gamma\epsilon^2}(\epsilon^4 + \epsilon^2/\gamma + 1/2\gamma^2) - e^{-2\gamma}(1 + 1/\gamma + 1/2\gamma^2)}{e^{-2\gamma\epsilon^2} - e^{-2\gamma}}. \tag{67}$$

Some typical numerical results are presented for $\gamma = 1$ in Figure 7 and Table 5.

It is evident from the data given in Tables 4 and 5 that the centroids of two PSF's for nonuniformly illuminated circular pupils aberrated by equal amounts of primary coma and secondary coma are different. For example, when $W_3 = W_5 = 1\lambda$, $\langle x \rangle_3 = 1.37$ and $\langle x \rangle_5 = 1.12$ where $\langle x \rangle$ is in units of λF. In the case of a uniformly illuminated circular pupil, $\langle x \rangle_3 = \langle x \rangle_5$ for $W_3 = W_5$ as may be seen from Tables 1 and 2. Of course, for an annular pupil, the centroids $\langle x \rangle_3$ and $\langle x \rangle_5$ for $W_3 = W_5$ are also different whether the pupil is uniformly or nonuniformly illuminated.



(a)

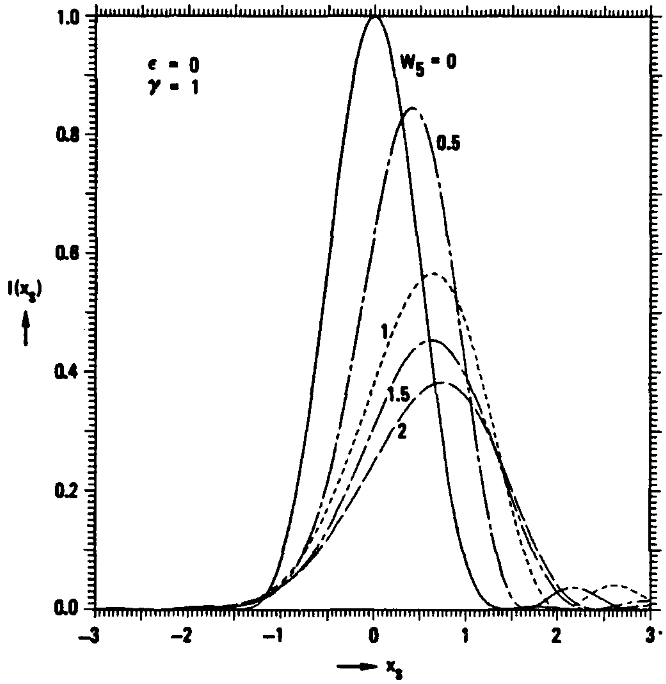


(b)

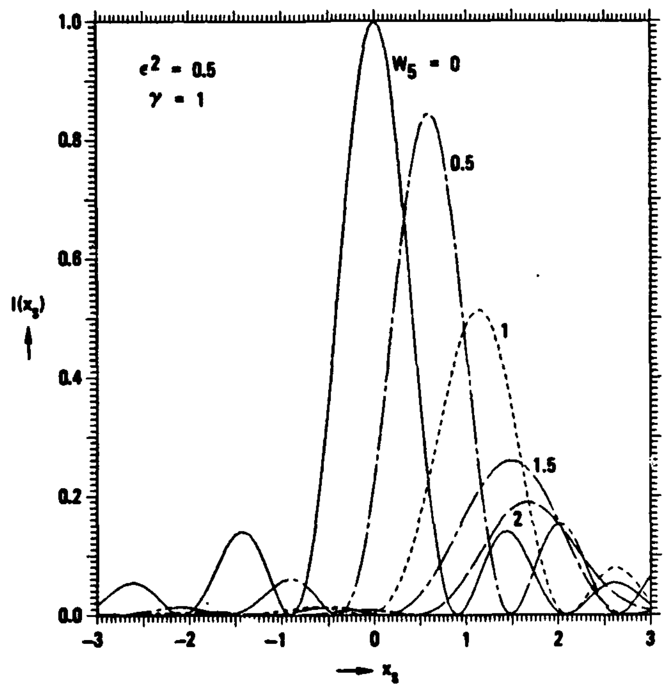
Figure 6. Same as Figure 1 except that the amplitude across the pupil is Gaussian given by Eq. (61) with $\gamma = 1$. The PSF's are normalized by the aberration- free value $I(0;1,\varepsilon)$ given by Eq. (63).

Table 4. Same as Table 1, except that $A(h)$ is a Gaussian given by Eq. (61) with $\gamma = 1$. The irradiances given here are normalized by the aberration-free central irradiance $I(0;1;\varepsilon)$ given by Eq. (63).

W_3	x_m	x_p	$\langle x \rangle$	I_m	I_p	I_c	$I(0)$
0	0	0	0	1.0000	1.0000	1.0000	1.0000
	(0)	(0)	(0)	(1.0000)	(1.0000)	(1.0000)	(1.0000)
0.50	0.61	0.60	0.69	0.8805	0.8806	0.8670	0.4567
	(0.76)	(0.76)	(1.42)	(0.9288)	(0.9288)	(0.1126)	(0.0602)
1.00	1.22	1.15	1.37	0.6013	0.6062	0.5590	0.1708
	(1.51)	(1.51)	(2.84)	(0.7435)	(0.7435)	(0.1273)	(0.0348)
1.50	1.82	1.40	2.06	0.3205	0.3672	0.2479	0.1199
	(2.27)	(2.24)	(4.25)	(0.5112)	(0.5122)	(0.0014)	(0.0033)
2.00	2.43	1.46	2.75	0.1305	0.2947	0.0624	0.0773
	(3.03)	(2.93)	(5.67)	(0.3005)	(0.3065)	(0.0399)	(0.0000)



(a)



(b)

Figure 7. Same as Figure 6, except that the aberration is secondary coma W_5.

Table 5. Same as Table 2, except that A(h) is a Gaussian given by Eq. (61) with $\gamma = 1$. The irradiances given here are normalized by the aberration-free central irradiance $I(0;1;\varepsilon)$ given by Eq. (63).

W_5	x_p	$\langle x \rangle$	I_p	I_c	$I(0)$
0	0	0	1.0000	1.0000	1.0000
	(0)	(0)	(1.0000)	(1.0000)	(1.0000)
0.50	0.41	0.56	0.8452	0.8105	0.6322
	(0.59)	(1.57)	(0.8451)	(0.0123)	(0.2253)
1.00	0.64	1.12	0.5659	0.4161	0.3793
	(1.14)	(3.14)	(0.5144)	(0.0026)	(0.0025)
1.50	0.63	1.68	0.4541	0.1147	0.3083
	(1.49)	(4.70)	(0.2595)	(0.0075)	(0.0011)
2.00	0.74	2.24	0.3824	0.0075	0.2476
	(1.67)	(6.27)	(0.1892)	(0.0043)	(0.0084)

V. DISCUSSION AND CONCLUSION

We have defined the LOS of an optical system in terms of the centroid of its PSF. Since the imaging properties of an optical system are determined by its pupil function, the centroid of the PSF is no exception. By expressing the centroid in terms of the pupil function, it is easy to show that the wave diffraction optics and ray geometrical optics give identical expressions for the centroid, regardless of the shape of the pupil or the amplitude and phase distributions across it. This is inspite of the fact that the two PSF's are quite different from each other. Although the LOS of an optical system can be obtained from the centroid of its PSF, or from the slope of the imaginary part of its OTF evaluated at the origin, in optical design and analysis, the simplest way to obtain the LOS would be to determine the centroid of the ray spot diagram. The idea is that, since ray tracing would be needed to calculate the aberrations of the system any way, one might as well trace the rays up to the image plane and calculate their centroid without calculating the diffraction PSF. Of course, the precision with which the centroid would be calculated would depend on the number of rays used, just as it would depend on the number of rays used to calculate the aberration function and the number of points used to calculate the aberrated PSF, for example, using an FFT algorithm.

In the case of an aberrated system with an annular pupil and a radially symmetric illumination, e.g., Gaussian, the LOS may be determined from the aberration coefficients of the appropriate orthogonal Zernike polynomials $S_n^1(\rho;\epsilon)\cos\theta$ and $S_n^1(\rho;\epsilon)\sin\theta$. When the amplitude across the pupil is uniform, these polynomials reduce to polynomials $R_n^1(\rho;\epsilon)\cos\theta$ and $R_n^1(\rho;\epsilon)\sin\theta$, respectively. The results for a circular pupil may be obtained by letting $\epsilon = 0$. In practice, however, given a certain optical system, the simplest way to determine the centroid would be to use its point spread function as measured, for example, by a photodetector array.

We have shown that for a uniformly illuminated pupil, since the centroid of an aberrated PSF depends only on the aberration along the perimeter of the pupil, aberrations such as $W_n \rho^n \cos\theta$ and $W_n R_n^1(\rho;0) \cos\theta$ (and similarly $W_n \rho^n \sin\theta$ and $W_n R_n^1(\rho;0) \sin\theta$) across a circular pupil with different aberration orders n, give aberrated PSF's with centroids that depend only on the value of W_n. Thus, for a given value of W_n, different aberration orders give PSF's with the same centroid.

We have obtained numerical results on the PSF's aberrated by primary and secondary coma for circular as well as annular pupils with uniform and Gaussian illuminations. In the case of uniform illumination, for a given value of W_3, the peak value, is higher for $\varepsilon^2 = 0.5$ than for $\varepsilon = 0$. The peak value as well as the centroid occur at larger value of x when ε is no zero compared to when it is zero. The peak value corresponds to minimum aberration variance ($I_p \sim I_m$) for $W_3 \lesssim 1.5\lambda$ when $\varepsilon = 0$ and for $W_3 \lesssim 2.5\lambda$ when $\varepsilon^2 = 0.5$. Except when $W_3 \sim 0$, I_c is quite small compared to I_p. As W_3 increases, $\langle x \rangle$ and x_m increase linearly with it. However, x_p first increases monotonically with W_3 and then fluctuates in a series of maxima and minima. For $\varepsilon^2 = 0.5$ the maxima and minima are widely spaced and are less pronounced compared to those for $\varepsilon = 0$.

In the case of secondary coma, the peak value does not occur for larger and larger values of x as W_5 increases, i.e., x_p does not increase monotonically with W_5. For example, when $\varepsilon = 0$, $x_p = 0.83$ for $W_5 = 1\lambda$ and $x_p = 0.81$ for $W_5 = 1.5\lambda$. Similarly, when $\varepsilon^2 = 0.5$, $x_p = 1.74$ when $W_5 = 1.5\lambda$ and $x_p = 1.71$ when $W_5 = 2\lambda$. As in the case of primary coma, $\langle x \rangle$ increases with ε for a given value of W_5.

In the case of Gaussian illumination numerical results are obtained for $\gamma = 1$. I_m and I_p are higher, and x_m, x_p and $\langle x \rangle$ are smaller for Gaussian illumination than those for uniform ($\gamma = 0$) illumination. It should be noted that I_m and I_p are normalized by the aberration-free peak irradiance, and it is only these normalized values that are higher for Gaussian illumination. [This normalization is different for Gaussian and uniform illuminations as may be seen by comparing Eqs. (41) and (63)]. Otherwise, for a given circular or annular aperture and for a given total power E, an unaberrated pupil gives maximum central irradiance when it is uniformly illuminated.[12]

The results given here are applicable to both imaging systems, e.g., those used for optical surveillance, as well as to laser transmitters used for active illumination of a target. In both cases, the LOS of the optical system is extremely important. A LOS error of a surveillance system will produce an error in the location of the target. In the case of a laser transmitter, a large LOS error may cause the laser beam to miss the target altogether. Whereas for static aberrations, we may be able to calibrate the LOS, for dynamic aberrations it is the analysis given here that will determine tolerances on aberrations of the type $\rho^n\cos\theta$ and $\rho^n\sin\theta$ in the case of annular or circular pupils.

Although we have defined the LOS of an optical system in terms of the centroid of its PSF, it could have been defined in terms of the peak of the PSF (assuming that the aberrations are small enough so that the PSF has a distinguishable peak). As pointed out in Section II, for an aberration-free PSF, its peak value and its centroid both lie at its origin, regardless of the amplitude variations across its pupil. The two are not coincident when aberrations are present. The precise definition of the LOS will perhaps depend on the nature of the application of the optical system. Moreover, in practice, only a finite central portion of the PSF will be sampled to measure its centroid, and the precision of this measurement will be limited by the noise characteristics of the photodetector array.

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APPENDIX. PSF for an Annular Pupil with Radially Symmetric Illumination and Coma Aberration

Consider an optical system having an annular pupil with radially symmetric illumination A(h) and coma aberration

$$W(h,\theta) = \sum_n{}' \rho^n (W_n \cos\theta + W'_n \sin\theta). \quad \text{(A1)}$$

where, as in Eq. (25), the prime on the summation sign indicates a summation over odd integral values of n. Note that $h = \rho a$ and $\varepsilon \leq \rho \leq 1$. In polar coordinates, Eq. (1) for the aberrated PSF may be written

$$I(r,\phi;\varepsilon) = (a^2/\lambda R)^2 \left| \int_\varepsilon^1 \int_0^{2\pi} A(h) \exp\{\pi i[2W(h,\theta) - r_s \rho \cos(\theta-\phi)]\} \rho \, d\rho \, d\theta \right|^2, \quad \text{(A2)}$$

where we have assumed that $W(h,\theta)$ and, therefore, W_n and W'_n are in units of λ. Integration over θ in Eq. (A2) can be carried out if we let

$$2W(h,\theta) - r_s \rho \cos(\theta-\phi) = B \cos(\theta-\psi), \quad \text{(A3)}$$

where

$$B^2 = (\sum_n{}' 2 W_n \rho^n - r_s \rho \cos\phi)^2 + (\sum_n{}' 2 W'_n \rho^n - r_s \rho \sin\phi)^2 \quad \text{(A4)}$$

and

$$\tan\psi = (\sum_n{}' 2W'_n\rho^n - r_s \sin\phi)/(\sum_n{}' 2W_n\rho^n - r_s \cos\phi) \quad \text{(A5)}$$

Thus, Eq. (A2) becomes

$$I(r,\phi;\varepsilon) = (\pi a^2/\lambda R)^2 \left[\int_{\varepsilon^2}^1 A(h) J_0(\pi B) dt\right]^2 , \quad \text{(A6)}$$

where we have let

$$t = \rho^2. \quad \text{(A7)}$$

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